Li

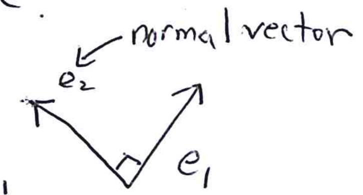
Tutorial 1 2015.09.10

1. Find the curvature of the curve given by the graph of a function $y = f(x)$.

Let's recall the definition of (signed) curvature $k(s)$ for a regular plane curve $c(s)$ which is parametrized by arc-length first.

We should assume $c(s) \in C^2$ otherwise we can not define curvature which is some kind of 2nd derivative quantity for the curve.

Tangent vector $T = e_1 = \frac{dc}{ds} = c'$, $e_2 \perp e_1$

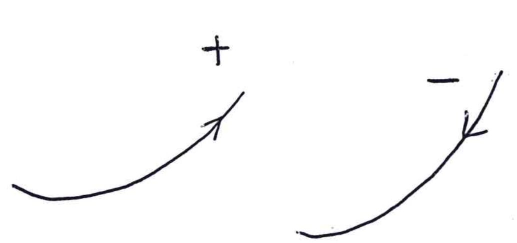



$$0 = \langle c', c' \rangle' = 2\langle c'', c' \rangle$$

$\Rightarrow$ $c'' = k e_2$, $k$ is some cts function $(=:\text{curvature})$

$k$ + : the tangent goes to the left

$k$ − : . . . . . . . . . . . right



$e_2 = (-y', x')$, since $e_1 = T = (x', y')$

$$K = \langle c'', e_2 \rangle = \langle (x'', y''), (-y', x') \rangle$$

$$= y'' \cdot x' - x'' \cdot y'$$

Now $c(t) = (t, f(t))$, $s(t) = \int_0^t \sqrt{1+\dot{f}^2}$, $\dot{s} = \sqrt{1+\dot{f}^2}$

$$x' = \frac{1}{\sqrt{1+\dot{f}^2}}, \quad y' = \frac{\dot{f}}{\sqrt{1+\dot{f}^2}}$$

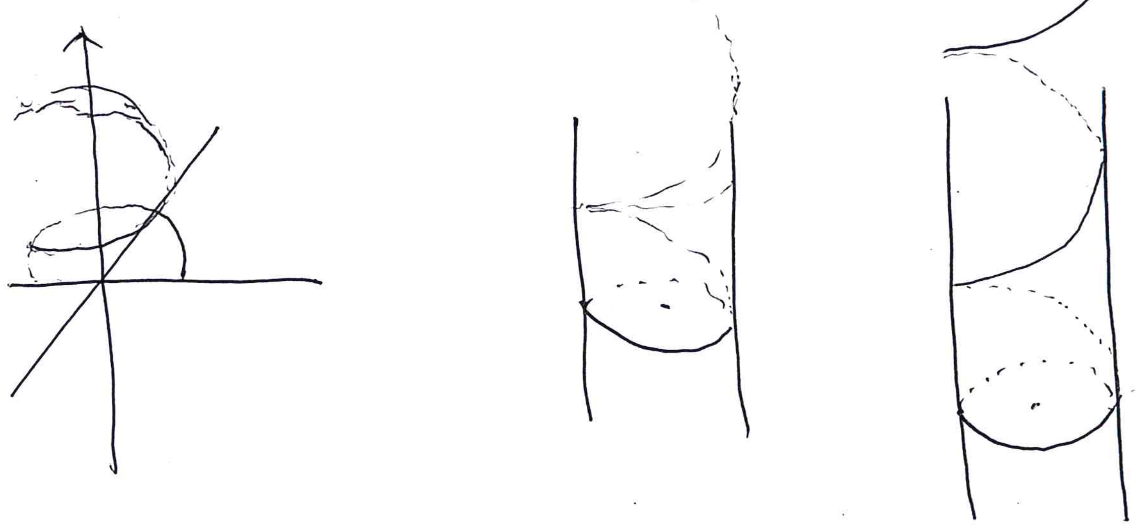
$$x'' = -\frac{1}{2} \frac{1}{(1+\dot{f}^2)^{\frac{3}{2}}} \cdot 2\dot{f} \cdot \ddot{f} \cdot \frac{1}{(1+\dot{f}^2)^{\frac{1}{2}}} = \frac{-\dot{f}\ddot{f}}{(1+\dot{f}^2)^2}$$

$$y'' = \frac{-\dot{f}^2 \ddot{f} + \ddot{f}(1+\dot{f}^2)}{(1+\dot{f}^2)^2} = \frac{\ddot{f}}{(1+\dot{f}^2)^2}$$

$$K = \frac{\ddot{f} + \text{[redacted]}\,\ddot{f}}{(1+\dot{f}^2)^{\frac{5}{2}}} = \frac{\ddot{f}}{(1+\dot{f}^2)^{\frac{3}{2}}}.$$

2. Calculate explicitly the curvature and torsion of a helix by doing arc-length parametrization.

$\beta(t) = (a\cos t,\ a\sin t,\ bt)$, $a>0$



$\dot{\beta} = (-a\sin t,\ a\cos t,\ b)$, $|\dot{\beta}| = \sqrt{a^2+b^2}\ (=: c)$

Then $s(t) = ct$.

$\tilde{\beta}(s) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{bs}{c}\right)$

$\kappa = |\tilde{\beta}''|$

$\tilde{\beta}' = \left(-\frac{a}{c}\sin\frac{s}{c},\ \frac{a}{c}\cos\frac{s}{c},\ \frac{b}{c}\right)$

$\tilde{\beta}'' = \left(-\frac{a}{c^2}\cos\frac{s}{c},\ -\frac{a}{c^2}\sin\frac{s}{c},\ 0\right)$

$\kappa = \frac{a}{c^2} > 0$.

$$\tau = -\langle B', N \rangle$$

$$N = \frac{T'}{\kappa} = \frac{\beta''}{\kappa} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$$

$$B = T \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{vmatrix}$$

$$= \left(\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right)$$

$$B' = \frac{b}{c^2}\left(\cos\frac{s}{c}, \sin\frac{s}{c}, 0\right) = -\frac{b}{c^2}N$$

$$\Rightarrow \tau = \frac{b}{c^2}$$

3. Local Canonical form of a curve (do Carmo)

$\alpha(s) = (\alpha^1(s), \alpha^2(s), \alpha^3(s))$ = unit speed curve

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$$\alpha' = T$$

$$\alpha'' = T' = \kappa N$$

$$\alpha''' = \kappa' N + \kappa N' = \kappa' N + \kappa(-\kappa T + \tau B)$$

$$= -\kappa^2 T + \kappa' N + \kappa\tau B.$$

Suppose the coordinate system of $\mathbb{R}^3$ is chosen s.t

$$\begin{cases} \alpha(0) = (0,0,0) \\ T(0) = e_1,\ N(0) = e_2,\ B(0) = e_3. \end{cases}$$

Taylor expansion $\Rightarrow$

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \alpha''(0)\frac{s^2}{2} + \alpha'''(0)\frac{s^3}{6} + o(s^3)$$

$$= sT(0) + \frac{s^2}{2}\kappa(0)N(0) + \frac{s^3}{6}\left(-\kappa^2(0)T(0) + \kappa'(0)N(0) + \kappa(0)\tau(0)B(0)\right)$$

$$+ \cdots$$

$$= \left(s - \frac{\kappa^2(0)}{6}s^3\right)e_1 + \left(\frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3\right)e_2 + \frac{\kappa(0)\tau(0)}{6}s^3 e_3 + \cdots$$

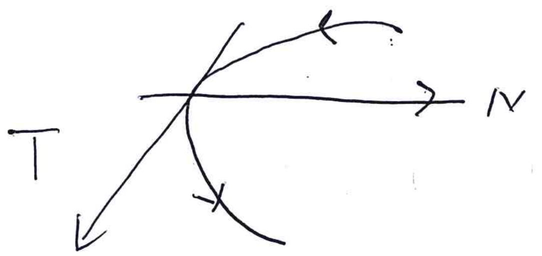
$$\Rightarrow \begin{cases} \alpha^1(s) = s - \frac{\kappa^2(0)}{6}s^3 + \cdots \\ \alpha^2(s) = \frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3 + \cdots \\ \alpha^3(s) = \frac{\kappa(0)\tau(0)}{6}s^3 + \cdots \end{cases} \qquad \cdots = o(s^3)$$

This form is called local canonical form of $\alpha$ in a neighborhood of $s=0$.

The projection ~~of~~ on TN-plane

$$x = \alpha^1(s) = s - \frac{\kappa^2(0)}{6} s^3 + \cdots$$
$$y = \alpha^2(s) = \frac{\kappa(0)}{2} s^2 + \frac{\kappa'(0)}{6} s^3 + \cdots$$

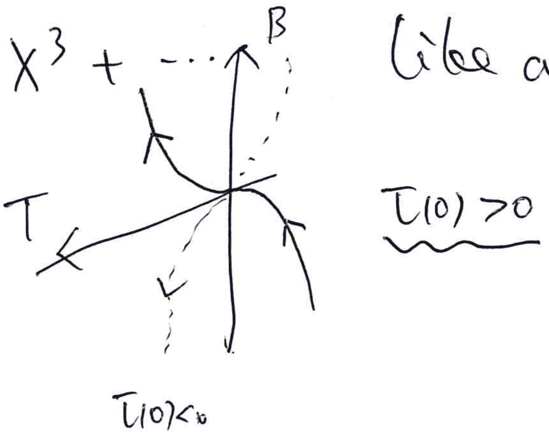
$$\Rightarrow \quad y = \frac{\kappa(0)}{2} x^2 + \cdots$$ like a parabola




The projection on TB-plane

$$\begin{cases} x = \alpha^1(s) = s - \dfrac{\kappa^2(0)}{6} s^3 + \cdots \\ z = \alpha^3(s) = \dfrac{\kappa(0)\,\tau(0)}{6} s^3 + \cdots \end{cases}$$

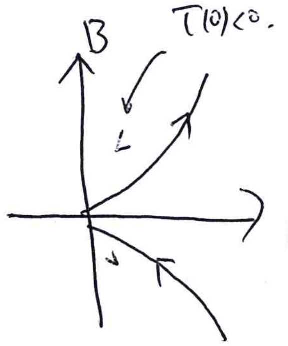
$$z = \frac{\kappa(0)\tau(0)}{6} x^3 + \cdots$$ like a cubic curve



Projection on NB-plane

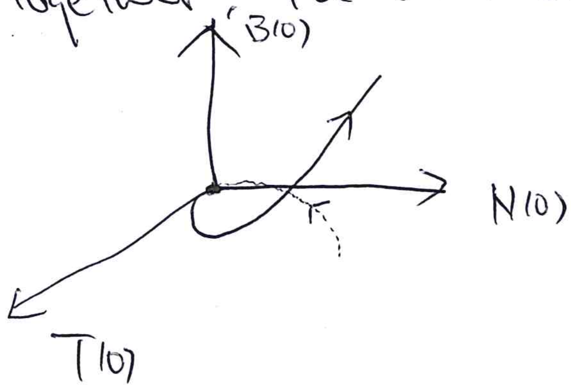
$$\begin{cases} y = \alpha^2(s) = \dfrac{\kappa(0)}{2} s^2 + \dfrac{\kappa'(0)}{6} s^3 + \cdots \\ z = \alpha^3(s) = \dfrac{\kappa(0)\tau(0)}{6} s^3 + \cdots \end{cases}$$

like a cusp $\sim (t^2, t^3)$




$\tau(0) > 0$

All together, the curve $\alpha$ looks like




$\tau(0) > 0$.